THEORY OF SQUARE-LIKE ABELIAN GROUPS IS DECIDABLE

OLEG BELEGRADEK

Abstract. A group is called square-like if it is universally equivalent to its direct square. It is known that the class of all square-like groups admits an explicit first order axiomatization but its theory is undecidable. We prove that the theory of square-like *abelian* groups is decidable. This answers a question posed by D. Spellman.

Introduction

A group G is called *discriminating* [1] if every group separated by G is discriminated by G. Here G is said to separate (discriminate) a group H if for any non-identity element (finite set of non-identity elements) of H there is a homomorphism from H to G which does not map the element (any element of the set) to the identity. A group G is discriminating iff G discriminates G^2 [1]. In particular, if G embeds G^2 then G is discriminating.

A group G is called *square-like* [5] if the groups G^2 and G are universally equivalent. Any discriminating group is square-like [4]. The notions of discriminating and square-like group were studied in [1, 3, 4, 5, 6, 7, 8, 9].

The class of square-like groups is first order axiomatizable [5], and the theory of the class is computably enumerable; an explicit first order axiom system was suggested in [2, 3], and also presented in [8]. In [5] square-like abelian groups were characterized in terms of Szmielew invariants.

The subclass of discriminating groups is not first order axiomatizable [5]. Every square-like group is elementarily equivalent to a discriminating group [3, 7]; so the class of square-like groups is the axiomatic closure of the class of discriminating groups.

The theory of square-like groups is undecidable [3, 7]. The argument in [7] is based on the obvious observation that any group embeds in a discriminating group, and so the universal theory of square-like groups coincide with the universal theory of all groups. The latter is undecidable because there exist finitely presented groups with unsolvable word problem. In [3] a discriminating group that interprets the ring of integers is constructed; any theory that has the group as a model (and, in particular, the theory of square-like groups) is undecidable.

Date: February 9, 2006.

2000 *Mathematics Subject Classification.* Primary: 20A15; Secondary: 20K99, 20E26, 03C60, 03D35.

The main result of the present paper is that the theory of square-like abelian groups is decidable. This answers a question posed by Dennis Spellman [12]. As a byproduct, we found characterizations of discriminating and square-like Szmielew groups.

1. Preliminaries

Here we collect some known definitions and facts we will use in the proofs.

Fact 1.1. [1, Proposition 1] *A group G is discriminating iff G discriminates G^2. In particular, G is discriminating if G embeds G^2.*

Fact 1.2. [1, Proposition 2] *The direct product* (*restricted or not*) *of any family of discriminating groups is a discriminating group.*

Fact 1.3. [1, Proposition 3] *Any torsion-free abelian group is discriminating.*

Fact 1.4. [4, Lemma 2.1] *Any discriminating group is square-like.*

Fact 1.5. [5, Theorem 3] *The class of square-like groups is first order axiomatizable.*

Fact 1.6. [3, Proposition 3.5] *Any* $\mathrm{End}(G)$*-invariant subgroup of a discriminating group G is trivial or infinite.*

Let A be an abelian group. For a positive integer n we denote

$$nA = \{na : a \in A\}, \quad A[n] = \{a \in A : na = 0\},$$

and write $\delta(A)$ for the largest divisible subgroup of A. We write $nA[k]$ for $(nA)[k]$. The subgroups nA, $A[n]$, $nA[k]$, and $\delta(A)$ are $\mathrm{End}(A)$-invariant. We write $A^{(\kappa)}$ for the direct sum of κ copies of A.

We write $\mathbb{Q}$ for the additive group of all rational numbers, and $\mathbb{Z}_{(p)}$ for the additive group of rational numbers with denominator not divisible by a prime p. We write $\mathbb{Z}(n)$ for the cyclic group of order n, and $\mathbb{Z}(p^\infty)$ for the Prüfer p-group.

A *Szmielew group* is defined to be an abelian group of the form

$$(\star) \qquad \bigoplus_{p \text{ prime}} [\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})} \oplus \mathbb{Z}(p^\infty)^{(\lambda_p)} \oplus \mathbb{Z}_{(p)}^{(\mu_p)}] \oplus \mathbb{Q}^{(\nu)}$$

where $\kappa_{p,n-1}$, λ_p, μ_p, ν are cardinals $\leq \omega$.

For a prime p, we call a Szmielew group of the form

$$\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})} \oplus \mathbb{Z}(p^\infty)^{(\lambda_p)} \oplus \mathbb{Z}_{(p)}^{(\mu_p)} \oplus \mathbb{Q}^{(\nu)}$$

a *p-Szmielew* group.

Fact 1.7. [11, Lemma A.2.3] *Every abelian group is elementarily equivalent to a Szmielew group.*

Let p be a prime, and $n, k < \omega$. Let $\Phi_k(p,n)$ and $\Phi^k(p,n)$ be the sentences that say about an abelian group B that

$$\dim_p(p^n B[p]/p^{n+1}B[p]) = k \quad \text{and} \quad \dim_p(p^n B[p]/p^{n+1}B[p]) > k,$$

$\Theta_k(p,n)$ and $\Theta^k(p,n)$ be the sentences that say that

$$\dim_p(p^n B[p]) = k \quad \text{and} \quad \dim_p(p^n B[p]) > k,$$

$\Gamma_k(p,n)$ and $\Gamma^k(p,n)$ be the sentences that say that

$$\dim_p(p^n B/p^{n+1}B) = k \quad \text{and} \quad \dim_p(p^n B/p^{n+1}B) > k,$$

$\Delta_k(p,n)$ and $\Delta^k(p,n)$ be the sentences that say that

$$|p^n B| = k \quad \text{and} \quad |p^n B| > k.$$

The sentences defined above are called the Szmielew invariant sentences. Note that $|B| = k$ and $|B| > k$ can be expressed as $\Delta_k(p,0)$ and $\Delta^k(p,0)$, for any prime p.

Fact 1.8. [11, Section A.2] *If A is the Szmielew group ($\star$) then*

- $A \models \Phi_k(p,n)$ *iff* $\kappa_{p,n} = k$,
- $A \models \Phi^k(p,n)$ *iff* $\kappa_{p,n} > k$,
- $A \models \Theta_k(p,n)$ *iff* $\lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \cdots = k$,
- $A \models \Theta^k(p,n)$ *iff* $\lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \cdots > k$,
- $A \models \Gamma_k(p,n)$ *iff* $\mu_p + \kappa_{p,n} + \kappa_{p,n+1} + \cdots = k$,
- $A \models \Gamma^k(p,n)$ *iff* $\mu_p + \kappa_{p,n} + \kappa_{p,n+1} + \cdots > k$.

Fact 1.9. [11, Theorem A.2.7] *Every sentence of the first order language of abelian groups is equivalent, modulo the theory of abelian groups, to a positive Boolean combination of Szmielew invariant sentences.*

Fact 1.10. [11, Theorem A.2.7] *Two abelian groups are elementarily equivalent iff they satisfy the same Szmielew invariant sentences.*

Abusing terminology, we call a sentence of the language of abelian groups *consistent* if it is true in some abelian group. By Fact 1.7, a sentence is consistent iff it holds in some Szmielew group.

Fact 1.11. [11, Theorem A.2.8] *There is an algorithm that, given a finite conjunction of Szmielew invariant sentences, decides whether it holds in some Szmielew group.*

Facts 1.9 and 1.11 are main ingredients of a proof of the Szmielew theorem on decidability of the theory of abelian groups; actually, they immediately imply the result. Indeed, given a sentence ϕ, by Fact 1.9 and computable enumerability of the theory of abelian groups, we can effectively find a positive Boolean combination θ of Szmielew invariant sentences that is equivalent to $\neg\phi$, modulo the theory. A sentence ϕ is not in the theory iff θ is consistent; the latter can be effectively checked, by Fact 1.11.

We will use a similar method in our proof of decidability of the theory of square-like abelian groups.

2. Discriminating and square-like Szmielew groups

Let A be the Szmielew group $(\star)$. For a prime p, let $I_p = \{n : \kappa_{p,n-1} > 0\}$. In case when the set I_p is finite and nonempty, l_p denotes its maximal element; clearly, $\kappa_{p,l_p-1} > 0$.

Proposition 2.1. *The following are equivalent:*

(1) *A is discriminating;*
(2) *for any prime p one of the following holds:*
    (i) *$\lambda_p = \omega$,*
    (ii) *$\lambda_p = 0$, and if I_p is finite and nonempty then $\kappa_{p,\,l_p-1} = \omega$.*

Proof. (1)⇒(2). Suppose (1). Let p be a prime. The subgroup $\delta(A) \cap A[p]$ is $\mathrm{End}(A)$-invariant, and hence is trivial or infinite, by Fact 1.6. Then λ_p is 0 or ω. Suppose $\lambda_p = 0$, and I_p is finite and nonempty. Then the $\mathrm{End}(A)$-invariant subgroup $p^{l_p-1}A[p]$ is nontrivial and hence infinite, again by Fact 1.6. Then $\kappa_{p,\,l_p-1} = \omega$.

(2)⇒(1). Suppose (2). Then for any prime p the group

$$\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})} \oplus \mathbb{Z}(p^\infty)^{(\lambda_p)}$$

embeds it square. So $A = B \oplus C$, where B embeds B^2, and C is torsion-free. By Facts 1.1, 1.3, and 1.2, A is discriminating. □

Proposition 2.2. *The following are equivalent:*

(1) *A is square-like;*
(2) *for any prime p one of the following holds:*
    (i) *$\lambda_p = \omega$,*
    (ii) *$\lambda_p = 0$, and if I_p is finite and nonempty then $\kappa_{p,\,l_p-1} = \omega$,*
    (iii) *$0 < \lambda_p < \omega$, and I_p is infinite.*

Proof. (1) ⇒ (2). Suppose (2) fails. Then, for some prime p, (i), (ii), (iii) all fail. There are two possibilities:

(a) $\lambda_p = 0$, the set I_p is finite, nonempty, and $\kappa_{p,\,l_p-1} < \omega$,
(b) $0 < \lambda_p < \omega$, and the set I_p is finite.

Suppose (a). Let $\kappa = \kappa_{p,\,l_p-1}$. We have

$$|p^{l_p-1}A[p]| = p^\kappa, \qquad |p^{l_p-1}A^2[p]| = p^{2\kappa}.$$

Suppose (b). Put $l = l_p$ if $I_p \neq \emptyset$, and $l = 0$ otherwise. We have

$$|p^l A[p]| = p^{\lambda_p}, \qquad |p^l A^2[p]| = p^{2\lambda_p}.$$

For any positive integers s and t there is an existential sentence that says about an abelian group B that $|sB[p]| \geq t$. Therefore in both cases (a) and (b) the groups A and A^2 are not universally equivalent, and so (1) fails.

(2) $\Rightarrow$ (1). Suppose (2). Let A' be the Szmielew group obtained from A by replacing

$$\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})} \oplus \mathbb{Z}(p^\infty)^{(\lambda_p)}$$

with

$$\bigoplus_{n>0} \mathbb{Z}(p^n)^{(\kappa_{p,n-1})},$$

for all p satisfying (3). Then A' is discriminating, by Proposition 2.1. Hence A' is square-like, by Fact 1.4. It is easy to check that A and A' satisfy the same Szmielew invariant sentences; therefore, by Fact 1.10, $A \equiv A'$. Then, by Fact 1.5, the group A is square-like, too. $\square$

Corollary 2.3. *Any square-like abelian group is elementarily equivalent to a discriminating Szmielew group.*

Proof. Let B be a square-like abelian group. By Fact 1.7, B is elementarily equivalent to a Szmielew group A. By Fact 1.5, A is square-like. The argument at the end of the proof of Proposition 2.2 shows that A is elementarily equivalent to a discriminating Szmielew group A'. $\square$

3. Main result

Theorem 3.1. *The theory of square-like abelian groups is decidable.*

Proof. We need to find an algorithm which, given a sentence ϕ of the language of abelian groups, decides whether ϕ is true in some square-like abelian group, or, equivalently by Corollary 2.3, in some discriminating Szmielew group. By Fact 1.9, ϕ is equivalent, modulo the theory of abelian groups, to a positive Boolean combination θ of Szmielew invariant sentences. Since the theory of abelian groups is computably enumerable, θ can be found effectively. We may assume that θ is $\bigvee_i \theta_i$, where each θ_i is a conjunction of finitely many Szmielew invariant sentences. So it suffices to prove

Claim. *There exists an algorithm that, given a consistent conjunction ψ of finitely many Szmielew invariant sentences, decides whether ψ holds in some discriminating Szmielew group.*

For a prime p, we call a conjunction of formulas of the forms

$$\Phi_k(p,n),\ \Theta_k(p,n),\ \Gamma_k(p,n), \Delta_k(p,n),$$
$$\Phi^k(p,n),\ \Theta^k(p,n),\ \Gamma^k(p,n),\ \Delta^k(p,n)$$

a p-conjunction. To prove the Claim, we show that

(A) *there exists an algorithm that, given a prime p and a consistent p-conjunction ψ, decides whether ψ holds in some discriminating p-Szmielew group*, and

(B) *the Claim follows from* (A).

First we show (B): assuming (A), we prove the Claim.

Let ψ be a conjunction of Szmielew invariant sentences, which holds in a Szmielew group A. We have $\psi = \bigwedge_p \psi_p$, where p runs over a finite set of primes, and ψ_p is a p-conjunction. There are three possibilities:

(a) ψ has no conjuncts of the form $\Delta_k(p, n)$;
(b) ψ has some conjuncts $\Delta_k(p, n)$ and $\Delta_l(q, m)$ with $p \neq q$;
(c) ψ has a conjunct $\Delta_k(p, n)$, but has no conjuncts $\Delta_l(q, m)$ with $p \neq q$.

The following three lemmas prove (B).

Lemma 3.2. *Assume* (a). *The following are equivalent:*

(i) *ψ holds in some discriminating Szmielew group,*
(ii) *for all p the sentence ψ_p holds in some discriminating p-Szmielew group.*

Proof. Suppose (i). We have $A = \oplus_p A(p)$, where $A(p)$ is a p-Szmielew group. Let p be a prime. Then $A(p) \oplus \mathbb{Q}$ is a discriminating p-Szmielew group, by Proposition 2.1. Also, $A(p) \oplus \mathbb{Q} \models \psi_p$ because of (a). So (ii) holds.

Suppose (ii). For every prime p choose a discriminating p-Szmielew group $A(p)$ in which ψ_p holds. By Proposition 2.1, the Szmielew group $A = \oplus_p A(p)$ is discriminating. For every p we have $A \models \psi_p$, because $A(p) \models \psi_p$ and ψ satisfies (a). Therefore $A \models \psi$. So (i) holds. □

Lemma 3.3. *Let B be a discriminating abelian group.*

(1) *If $\Delta_k(p, n)$ or $\neg\Delta^k(p, n)$ holds in B then $p^n B = 0$.*
(2) *Assume* (b). *If $B \models \psi$ then $B = 0$.*

Proof. (1) The subgroup $p^n B$ is $\mathrm{End}(B)$-invariant and finite of order at most k. By Fact 1.6, the result follows.

(2) By (1), $p^n B = q^m B = 0$, and hence $B = 0$. □

Thus, for any ψ with (b), in order to decide whether there is a discriminating Szmielew group that satisfies ψ, we need to decide whether ψ holds in the trivial group, which can be done effectively.

Lemma 3.4. *Assume* (c). *Then ψ holds in some discriminating Szmielew group if and only if*

(i) *For any $q \neq p$ and $l > 0$, in ψ there are no conjuncts of the forms*
$$\Phi^l(q, m),\ \Theta^l(q, m),\ \Gamma^l(q, m),\ \Phi_l(q, m),\ \Theta_l(q, m),\ \Gamma_l(q, m);$$
(ii) *For any $q \neq p$, in ψ there are no conjuncts of the forms*
$$\Phi^0(q, m),\ \Theta^0(q, m),\ \Gamma^0(q, m);$$
(iii) *the p-conjunction*
$$\psi_p \wedge \bigwedge\{\Delta^s(p, 0) : s \in S\}$$
holds in some discriminating p-Szmielew group, where S is the set of all s such that $\Delta^s(q, m)$ is a conjunct of ψ, for some $q \neq p$ and some m.

Proof. First suppose that ψ holds in a discriminating Szmielew group A. By (c) and Lemma 3.3 (1), $p^n A = 0$, and so A is a p-Szmielew group. Therefore (i) and (ii) hold. Let $s \in S$. Then for some m and $q \neq p$ we have $A \models \Delta^s(q,m)$, that is, $|q^m A| > s$. As $p^n A = 0$, we have $q^m A = A$; thus $|A| > s$. Then $A \models \Delta^s(p,0)$. So (iii) holds.

Now suppose (i)–(iii) hold. By (iii) there is a discriminating p-Szmielew group A in which ψ_p and $\{\Delta^s(p,0) : s \in S\}$ are true. We show that $A \models \psi$. Since $\Delta_k(p,n)$ is a conjunct of ψ, we have $p^n A = 0$, by Lemma 3.3 (1). As A is a p-Szmielew group, all the sentences $\Phi_0(q,m)$, $\Theta_0(q,m)$, $\Gamma_0(q,m)$ with $q \neq p$ hold in A. Due to (i) and (ii), it remains to show that if $\Delta^s(q,m)$ is a conjunct of ψ, where $q \neq p$, then it holds in A. Suppose not. Then $q^m A = 0$, by Lemma 3.3 (1). Therefore $A = 0$, contrary to $A \models \Delta^s(p,0)$. □

Now we prove (A). From now on, *let p be a fixed prime, and ψ be a p-conjunction which holds in some Szmielew group A.* We will show how to decide whether ψ holds in some discriminating p-Szmielew group.

There are four possibilities:

(a) ψ has a conjunct $\Delta_k(p,n)$ with $k \neq 1$;
(b) ψ has a conjunct $\Theta_k(p,n)$ with $k > 0$;
(c) ψ has no conjuncts of the forms $\Delta_k(p,n)$ and $\Theta_k(p,n)$;
(d) ψ has a conjunct $\Delta_1(p,n)$ or $\Theta_0(p,n)$, but (a) and (b) fail.

Lemma 3.5. *If* (a) *then ψ fails in every discriminating abelian group.*

Proof. Suppose ψ holds in an abelian group B. Then $|p^n B| = k \neq 1$, and so $p^n B$ is a nontrivial finite $\mathrm{End}(B)$-invariant subgroup. Therefore B is not discriminating, by Fact 1.6. □

Lemma 3.6. *If* (b) *then ψ fails in every discriminating Szmielew group.*

Proof. Suppose $A \models \psi$, and A is a discriminating Szmielew group. Then

$$\omega > k = \lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \dots .$$

Hence $\lambda_p < \omega$ and so, by Proposition 2.1, $\lambda_p = 0$. Then

$$0 < \kappa_{p,n} + \kappa_{p,n+1} + \dots < \omega,$$

and so I_p is finite. Then we have $n < l_p$, and $\kappa_{p,l_p-1} < \omega$. In this case A is not discriminating, by Proposition 2.1. A contradiction. □

Lemma 3.7. *If* (c) *then ψ holds in some discriminating p-Szmielew group.*

Proof. We have $A = \oplus_q A(q)$, where $A(q)$ is a q-Szmielew group. Put

$$A'(p) := A(p) \oplus \mathbb{Z}(p^\infty)^{(\omega)}.$$

By Proposition 2.1, $A'(p)$ is a discriminating p-Szmielew group. Moreover, $A'(p) \models \psi$. Indeed, for any sentence θ of one of the forms

$$\Phi_k(p,n),\ \Phi^k(p,n),\ \Theta^k(p,n),\ \Gamma_k(p,n),\ \Gamma^k(p,n),\ \Delta^k(p,n)$$

if $A \models \theta$ then $A'(p) \models \theta$. □

It remains to consider case (d). We will need

Lemma 3.8. *For any $n \geq k$ the sentence $\Gamma_l(p,k)$ is effectively equivalent in abelian groups to a positive Boolean combination of sentences of the forms $\Gamma_i(p,n)$ and $\Phi_j(p,s)$, where $k \leq s < n$ and $0 \leq i, j \leq l$.*

Proof. It suffices to show that in abelian groups $\Gamma_l(p,k)$ is equivalent to

$$\Gamma'_l(p,k) := \bigvee_{i=0}^{l} (\Gamma_{l-i}(p,k+1) \wedge \Phi_i(p,k)).$$

A Szmielew group A satisfies $\Gamma_l(p,k)$ if and only if

$$\mu_p + \kappa_{p,k} + \kappa_{p,k+1} + \cdots = l;$$

the latter holds if and only if, for some $i \in \{0,1,\ldots,l\}$,

$$\mu_p + \kappa_{p,k+1} + \kappa_{p,k+2} + \cdots = l - i \quad \text{and} \quad \kappa_{p,k} = i,$$

which means that $\Gamma'_l(p,k)$ holds in A. □

Let $n < \omega$ be given. Replace in ψ every conjunct $\Gamma_l(p,k)$, where $k < n$, with an equivalent positive Boolean combination of sentences of the forms $\Gamma_i(p,n)$ and $\Phi_j(p,s)$. The resulting formula is equivalent to a disjunction of p-conjunctions in each of which there is no conjunct $\Gamma_l(p,k)$ with $k < n$. Therefore it remains to prove the following statement, which allows to decide whether ψ holds in some discriminating p-Szmielew group, in case (d).

Lemma 3.9. *Suppose that ψ has*
- (a) *a conjunct $\Delta_1(p,n)$ or $\Theta_0(p,n)$;*
- (b) *no conjuncts $\Delta_k(p,m)$ with $k \neq 1$ and $\Theta_k(p,m)$ with $k > 0$;*
- (c) *no conjuncts $\Gamma_l(p,s)$ with $s < n$.*

Then the following are equivalent:
- (1) *ψ fails in any discriminating p-Szmielew group;*
- (2) *there exist m with $m < n$ and $i > 0$ such that*
  - (i) *$\Phi_i(p,m)$ is a conjunct of ψ,*
  - (ii) *for every k with $m < k < n$ there is j such that $\Phi_j(p,k)$ is a conjunct of ψ.*

Proof. First we show that (b) implies that ψ holds in some p-Szmielew group. If $\Delta_1(p,n)$ is in ψ then $p^n A = 0$; therefore A is a direct sum of cyclic p-groups and hence a p-Szmielew group. Suppose $\Delta_1(p,n)$ is not in ψ. Let $A = \oplus_q A(q)$, where each $A(q)$ is a q-Szmielew group. Since ψ is a p-conjunction without conjuncts of the form $\Delta_k(p,n)$, the p-Szmielew group $A(p) \oplus \mathbb{Q}$ satisfies ψ.

So we may assume that A is a p-Szmielew group. By (a),

$$\lambda_p = \kappa_{p,n} = \kappa_{p,n+1} \cdots = 0.$$

Indeed, if $\Delta_1(p,n)$ is in ψ then $p^n A = 0$; if $\Theta_0(p,n)$ is in ψ then

$$0 = \lambda_p + \kappa_{p,n} + \kappa_{p,n+1} + \ldots .$$

In particular, the set I_p is finite.

Suppose (2). Due to (i), we have $\kappa_{p,m} = i > 0$, and therefore $m < l_p \le n$. Let $m < k < n$. By (ii) ψ has a conjunct $\Phi_j(p,k)$; then $\kappa_{p,k} = j$. So $\kappa_{p,k} < \omega$ for all k with $m \le k < n$. In particular, $\kappa_{p,l_p-1} < \omega$. By Proposition 2.1, in this case A cannot be discriminating, and (1) follows.

Assuming that (2) is not true, we show that (1) is not true, too.

If $I_p = \emptyset$ then A itself is discriminating, by Proposition 2.1.

Suppose $I_p \ne \emptyset$. First we show that there is $k < n$ such that $\kappa_{p,r} = 0$ for $r > k$, and for every j the sentence $\Phi_j(p,k)$ is not a conjunct of ψ. Let $m = l_p - 1$ and $i = \kappa_{p,m}$. Then $m < n$ and $i > 0$. If (i) fails, put $k := m$. If (i) holds then (ii) fails, and therefore there is k with $m < k < n$ such that for every j the sentence $\Phi_j(p,k)$ is not a conjunct of ψ.

By Proposition 2.1, the p-Szmielew group $A \oplus \mathbb{Z}(p^{k+1})^{(\omega)}$ is discriminating. Moreover,

$$A \oplus \mathbb{Z}(p^{k+1})^{(\omega)} \models \psi.$$

Indeed, by (c) and the choice of k, a conjunct θ of ψ can have only the forms

$$\Phi_j(p,r),\ \Theta_0(p,n),\ \Gamma_j(p,s),\ \Delta_1(p,n),$$

where $r \ne k$ and $s \ge n$, or the forms

$$\Phi^j(p,t),\ \Theta^j(p,t),\ \Gamma^j(p,t),\ \Delta^j(p,t).$$

Therefore $A \models \theta$ implies $A \oplus \mathbb{Z}(p^{k+1})^{(\omega)} \models \theta$, for all such θ. Here we use that $s \ge n > k$ when consider θ of the forms $\Theta_0(p,n)$ and $\Gamma_j(p,s)$. □

The proof of Theorem 3.1 is completed. □

4. Open questions

Proposition 4.1. *The theory of square-like nilpotent groups is undecidable.*

Proof. In fact, even the universal theory of square-like nilpotent groups is undecidable. Indeed, it coincides with the universal theory of nilpotent groups because any nilpotent group G embeds in the discriminating nilpotent group G^ω. As any finitely generated nilpotent group is residually finite, the universal theory of nilpotent groups coincides with the universal theory of finite nilpotent groups. The latter is undecidable [10]. □

Question. *Is the theory of square-like* 2*-step nilpotent groups undecidable?*

Note that the *universal* theory of square-like 2-step nilpotent groups is decidable. Indeed, as above, it coincides with the universal theory of 2-step nilpotent groups and with the universal theory of finite 2-step nilpotent groups. Obviously, the universal theory of 2-step nilpotent groups is computably enumerable, and the universal theory of finite 2-step nilpotent groups is co-computably-enumerable; so the result follows.

Thus, undecidability of the theory of square-like 2-step nilpotent groups cannot be shown like in the proof of Proposition 4.1. In [3, Theorem 5.1] we

proved undecidability of the theory of square-like groups by constructing a discriminating group which interprets the ring of integers.

Question. *Is there a discriminating* 2*-step nilpotent group which interprets the ring of integers?*

Existence of such a group would imply undecidability of the theory of square-like 2-step nilpotent groups.

References

[1] G. Baumslag, A. G. Myasnikov and V. N. Remeslennikov, Discriminating and co-discriminating groups, *J. Group Theory* **3** (2000), 467–479.
[2] O. Belegradek, Review of [5], *Math. Reviews*, MR1914831 (2003d: 20003).
[3] O. Belegradek, Discriminating and square-like groups, *J. Group Theory* **7** (2004), 521–532.
[4] B. Fine, A. M. Gaglione, A. G. Myasnikov and D. Spellman, Discriminating groups, *J. Group Theory* **4** (2001), 463–474.
[5] B. Fine, A. M. Gaglione, A. G. Myasnikov and D. Spellman, Groups whose universal theory is axiomatizable by quasi-identities, *J. Group Theory* **5** (2002), 365–381.
[6] B. Fine, A. M. Gaglione, D. Spellman, Every abelian group universally equivalent to a discriminating group is elementarily equivalent to a discriminating group, in *Combinatorial and geometric group theory*, Contemp. Math. **296** (Amer. Math. Soc., Providence, RI, 2002), 129–137.
[7] B. Fine, A. M. Gaglione, D. Spellman, The axiomatic closure of the class of discriminating groups, *Arch. Math.* **83** (2004), 106–112.
[8] B. Fine, A. M. Gaglione, D. Spellman, Discriminating and square-like groups. I. Axiomatics, in *Groups, statistics and cryptography*, Contemp. Math. **360** (Amer. Math. Soc., Providence, RI, 2004), 35–46.
[9] B. Fine, A. M. Gaglione, D. Spellman, Discriminating and square-like groups. II. Examples, *Houston J. Math.* **31** (2005), 649–674.
[10] O. G. Kharlampovich, Universal theory of the class of finite nilpotent groups is undecidable, *Math. Notes* **33** (1983), 254–263.
[11] W. Hodges, *Model theory*, Cambridge University Press, 1993.
[12] D. Spellman, *Private communication*, March 14, 2005.

Department of Mathematics, Istanbul Bilgi University, 80370 Dolapdere–Istanbul, Turkey
E-mail address: `olegb@bilgi.edu.tr`